MICROWAVE BACKGROUND: TEMPERATURE FLUCTUATIONS

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ABSTRACT. This second lecture on the μ-wave background radiation develops the physics of temperature fluctuations. I give a rough overview of cosmological perturbation theory, a kinetic theory approach to radiation transport, and the modes of excitation in the Universe around the time of decoupling. I then show how this leads to the observed temperature anisotropy. A later lecture will cover the phenomenology and current status of temperature anisotropy measurements.

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1. INTRODUCTION

In the first μ-wave background lecture, it was shown how the decoupling of radiation from matter leaves a relic radiation spectrum which is very close to thermal over a large range of frequencies. Decoupling began at a temperature of about 0.28 eV, or about 3500 K, at a redshift of roughly 1300, and the evolution of the ionization was certainly complete by the time the temperature dropped to 0.20 eV.

The discussion of decoupling in that lecture was one-dimensional, treating the evolution of one packet of radiation and the gas to which it was coupled, as the photons evolved along a line of sight to an observation point at the present time. But of course, an observer at that point can measure along any line of sight, observing any point on the projected sky. Since our basic picture of structure formation in the Universe requires the presence of matter/energy fluctuations, we should expect to see some differences between the measured temperatures along different lines of sight. Such differences are observed, at the level of $\Delta T/T \simeq 10^{-5}$. These fluctuations in the temperature on the sky give a more-or-less direct measure of the amplitude and correlations of the matter/energy fluctuations present in the Universe at the time of decoupling.

Over the last ten years, temperature fluctuations in the microwave background have become a focus for precision studies of cosmology. The technical reasons for this will be explained below. We will have more to say about the history of the search for temperature fluctuations when discussing the details of the observations, mostly in the third lecture.

The subject of this lecture is quite technical, but one should at least take from it some notions of how photons couple to baryons, what types of excitations occur and what different types of initial conditions are possible, how the primordial fluctuations leave direct imprints in the low order and local correlations of the photons, how anisotropies grow in free streaming, and how these become projected onto the sky

In the following we will develop those parts of cosmological perturbation theory which we need to discuss the evolution of fluctuations in matter and radiation. The evolution of perturbations along a given line of sight will be codified in the so-called line-of-sight integrals. Finally we will show how these line-of-sight integrals lead to sky multipoles.

2. Cosmological Perturbation Theory

2.1. **Orientation.** The microwave background is an essentially unambiguous probe of the early evolution of cosmological perturbations because everything of interest happens in the linear regime, where the theory is easily controlled. So the first thing we need to do is develop a theoretical picture of this linearized evolution. Some of this has been covered in previous lectures on density perturbations, in this cosmology lecture series.

Much of what will be said in the following is common knowledge about kinetic theory and fluid dynamics, simply applied to the cosmological case.

2.2. **Metric Perturbations.** Because of the usual gauge freedom in general relativity, there are many ways to write the perturbed metric. In the so-called synchronous gauge (which is not really a gauge choice at all!), the metric is taken to have the form

$$ds^2 = dt^2 - (a(t)^2\delta_{ij} + h_{ij})dx_i dx_j.$$

In the so-called conformal Newtonian gauge, the metric is taken to have the form

$$ds^2 = a(\eta)^2\left[(1+2\psi)d\eta^2 - (1-2\phi)d\vec{x}^2\right].$$

This form is reasonably close to one particular gauge-covariant formalism, and it is a true gauge choice; many calculations have been done with this gauge choice [MB95, HS94].

A physically transparent gauge-covariant formalism has been developed by Challinor and Lasenby [CL98]. This gauge-covariant formalism gives a coherent picture of perturbations around backgrounds other than spatially flat FRW, including the various multipole expansions which arise. It is based on a method for covariant perturbation theory introduced by Ellis and Bruni.

To the extent that we can do so in this limited exposition, we will follow this gauge-covariant approach. Unfortunately, it will not be possible to derive everything from scratch. When it simplifies the discussion we will always specialize to the case of a spatially flat FRW background. This should be adequate for the armchair cosmologist; those interested in full details are encouraged to consult the Challinor and Lasenby reference, which provides adequate references for the full development of the subject.

Throughout the discussion of cosmological perturbations, there is an implicit split of all quantities, in the form $Q = Q_0 + \epsilon Q_1$. The zeroth order quantities are those given by the fixed background, and the formal parameter ϵ controls the perturbative expansion of all quantities. The formal parameter will usually not appear; this is not a big problem for perturbation around FRW solutions since only a small number of distinct quantities appear in such backgrounds, and it is easy to keep track of them. Anything else is automatically a first order quantity.

The formalism first requires a choice of a fixed timelike vector field of unit norm u^a, throughout spacetime. This is equivalent to a choice of local observers. The only requirement on u^a is that it reduce to the FRW observers at zeroth order in perturbation theory. This means that it has the formal expansion $u_a = (dt)_a + \epsilon u'_a$; this expansion will never appear explicitly, as we deal only with u_a itself. Though it may at first seem odd to make such a choice in a gauge-covariant formulation, it is actually very natural, since it provides physical interpretations for all the quantities that enter. In particular, the hydrodynamic and kinetic variables have fairly obvious meanings.

Given the vector field u^a, the metric can be decomposed as $g_{ab} = u_a u_b + h_{ab}$. This defines the spatial metric h_{ab}, which is chosen to have the properties of a projection onto the spacelike slices orthogonal to u^a, $h_{ab}u^a = 0$, $h_{ab}h^b_c = h_{ac}$. Introduce $\dot{f} = u^b \nabla_b f$ for variations in the timelike direction, and introduce a derivative on spatial slices by $D_a t_{b_1 \cdots b_n} = h^c_a \nabla_c t_{b_1 \cdots b_n}$. The derivative of the vector field u can be decomposed in the useful form

$$\nabla_a u_b = \tilde{\omega}_{ab} + \sigma_{ab} + \frac{1}{3}\theta h_{ab} + u_a w_b,$$

where $w_a = u^b \nabla_b u_a$ is the acceleration, $\theta = \nabla^a u_a$ is called the expansion, $\tilde{\omega}_{ab} = D_{[a}u_{b]}$ is the vorticity, and $\sigma_{ab} = D_{(a}u_{b)}$ is the shear. This is just an identity, involving no assumptions; it has been seen before, in the first lecture of the series. Of the quantities on the right hand side, only the expansion is zeroth order in cosmological perturbation theory; the other terms are first order. The main reason we introduce this identity is to recall the nature of the expansion, θ, which describes how neighbouring integral curves of the vector field diverge from each other. So θ is really the local measure of the cosmological expansion. In fact it measures the local expansion of volumes, and we could write $\theta = 3H$, where H is the local Hubble expansion. Since θ is non-vanishing at zeroth order in the perturbation, it appears directly in many of the matter evolution equations.

Given the expansion factor θ it is also useful to introduce a local scale factor S. We can constructively define such a scale factor by

$$S(x) = \exp\left(\frac{1}{3}\int_{x_0}^{x} \nabla_a \theta\right).$$

Then $\dot{S}/S = \theta/3 = H$. Note that in general S varies along spatial sections, but in the formal perturbation theory in ϵ, it must be higher order since $D_a\theta$ vanishes in the FRW limit. This spatial variation is interesting in itself, and we introduce the co-moving expansion gradient

$$\mathcal{Z}_a = SD_a\theta.$$

The stress tensor is decomposed in a 3+1 form as well. We write

$$T_{ab} = \rho u_a u_b + u_a q_b + u_b q_a - p h_{ab} + \pi_{ab},$$

where $\rho = T_{ab}u^a u^b$ is the energy density measured by the u_a observer, $q_a = h_a^b T_{bc} u^c$ is the energy flux, $p = -h^{ab}T_{ab}/3$ is the pressure (the sign appears because of the metric convention), and $\pi_{ab} = h^{ab}T_{ab} + p h_{ab}$ is called the anisotropic stress. We will see below how these quantities can be calculated for particular forms of matter.

Given the above definitions, one can write down the equations for the evolution of gravitational perturbations. They are somewhat long, and we will not need them in what follows, although they are certainly needed if one intends to actually solve the equations. They are given in the references [CL98, BDE9X].

As an aside, note that the only nontrivial geometric equation for the background FRW, which is often written as

$$\dot{H} + H^2 + \alpha(\rho + 3p) = 0,$$

has an expression in terms of the expansion,

$$\dot{\theta} + \frac{1}{3}\theta^2 + 3\alpha(\rho + 3p) = 0.$$

This is an equation we saw in the first lecture as well; it is the Raychaudhuri equation.

2.3. **Linearized Kinetic Theory.** Conceptually, the easiest form of matter to deal with is particles. The particle species may be free streaming or they may have pointlike interactions amongst themselves. We can deal with this kind of matter by using kinetic theory; in the case of pointlike interactions at sufficiently low density, this means applying the Boltzmann equation. In relativistic kinetic theory, the phase space description is local, but otherwise the formalism is completely analogous to the non-relativistic case. This type of description is especially important for the massless species, photons and neutrinos. So we will concentrate on the massless case here.

The local momentum of a massless particle will be written $p_a = E(u_a + e_a)$, where E is the energy in the u_a observer frame and e_a is a spacelike unit vector with $u \cdot e = 0$. Think of e_a as a set of direction cosines which locally give the direction that the particle is traveling. Let $f(x, p)$ be the phase-space distribution function for the particles; we can also write it as $f(E, e)$.

Then we can sum up the contributions to the stress-energy tensor from all the particles at a point to get

$$T_{ab} = \int E^2 dE d\Omega(e) f(E,e) \frac{p_a(E,e)p_b(E,e)}{E}.$$

Connecting this with our previous decomposition of the stress tensor gives

$$\begin{aligned}
\rho &= \int E^3 dE d\Omega(e) f(E,e), \\
p &= \int E^3 dE d\Omega(e) f(E,e) \frac{1}{3} = \frac{1}{3}\rho, \\
q_a &= \int E^3 dE d\Omega(e) f(E,e) e_a, \\
\pi_{ab} &= \int E^3 dE d\Omega(e) f(E,e) e_a e_b + p h_{ab}.
\end{aligned}$$

These are the only moments which enter into the Einstein equations, since this is the full content of the stress tensor. But of course, there are an infinite number of higher moments. All of these moments evolve, and they are generally coupled to each other. To see how they evolve we will apply the Boltzmann equation. The Boltzmann equation always has the form

$$L_B f = C(f),$$

where L_B is the part which describes free particle evolution and $C(f)$ is some functional describing the rearrangement of particles due to short-range interactions (collisions). For our applications, we will only be interested in the case where the particles scatter off of a fixed background. This will make the equation easy to deal with since it will be linear.

First we should define L_B. L_B is the generator of free (geodesic) motion in phase space, which for a single particle is given by

$$\begin{aligned}
\frac{dx_\mu}{d\lambda} &= p_\mu, \\
p^a \nabla_a p_b &= 0,
\end{aligned}$$

where λ is an affine parameter along the null geodesic. So we have

$$L_B f = \frac{d}{d\lambda} f(x(\lambda), p(\lambda)).$$

As a mathematical aside, note that this thing is not a Lie derivative, because f is not a function of position alone. Actually f is a function on the cotangent bundle, which is identified as phase space, and L_B defines a second order differential equation on the manifold.

So far this is mostly just a bunch of definitions. The physical content resides in the collision operator $C(f)$. The two massless species of interest to us are neutrinos and photons. Neutrinos are easy; for our purposes they are noninteracting, so we write $C(f) = 0$ for neutrinos. For photons we must consider scattering from charged particles. Recall our discussion of photon scattering from the first lecture. In the low-energy regime where we are working, $T \ll m_e$, we can think of the scattering as occurring by different processes. There is Thomson scattering, which changes the direction of the photons but does not change their energy. There is Compton scattering, which can change photon energies but which conserves photon number; Compton scattering is suppressed relative to Thomson scattering by a factor of E/m_e. And finally there are processes which are relevant only at much higher energies, which can fully scramble the photon distribution.

In principle we must decide which of these processes are important for us. Recall the discussion in the first μ-wave background lecture. We know that Thomson scattering cannot produce a thermalization. Even Compton scattering cannot produce true thermalization, since it conserves photon number. So if we are really interested in the hydrodynamic limit for the radiation, with approximate local equilibrium, we would have to do alot of work. Is this what we need to do?

This question was implicitly answered at the end the first μ-wave background lecture, where we calculated the transparency of the Universe. Thomson scattering makes the Universe foggy, and the optical depth for this kind of fogginess was calculated there. It is this fog which interests us because that is what will distort our picture of the spatial structure of the photon spectrum. But it cannot distort the energy structure of the spectrum, so by limiting ourselves to a consideration of Thomson scattering we are implicitly limiting the types of effects that we can describe. In particular, as a matter of principle you cannot describe thermalization using only Thomson scattering, and therefore you cannot describe the usual form of the hydrodynamic limit. Later we will do some manipulations and call it radiation hydrodynamics, but this caveat is always understood.

This said, we will now go on to consider the evolution of photons using only Thomson scattering. Electrons dominate the scattering of photons since they allow the highest momentum transfer. The electron density tracks the ion density precisely, so we will speak of electrons and baryons interchangeably. In the frame of the scatterer, the differential cross-section for Thomson scattering looks like

$$\frac{d\sigma}{d\theta} \propto \sigma_T \left(1 + \cos^2\theta\right) .$$

We can write this in covariant form by noting that the cosine of the angle between the incoming and outgoing photon is given by $g^{ab}n_a(\mathrm{in})n_b(\mathrm{out})$, where n is a spacelike photon direction vector, orthogonal to the electron velocity, since the non-covariant result above is implicitly given in the frame where the electron is at rest. The flux factor can be written covariantly as $p^a v_a$, where v_a is the electron 4-velocity. So the collision term for photons can be written

$$C(f) = n_e \sigma_T p^a v_a \left[\frac{3}{16\pi} \int d\Omega(n') f(E, n') \left(1 + (n \cdot n')^2\right) - f(E, n) \right].$$

Carefully keeping track of the various relative velocities, using our perturbative expressions for geometric quantities that appear, and integrating over energies, we can obtain the Boltzmann equation in the following form.

$$\begin{aligned} \int E^2 dE L_B f(E, e) = & - n_e \sigma_T \int E^3 dE f(E, e) \\ & + \frac{3}{16\pi} n_e \sigma_T \left[\frac{4}{3} \left(1 - 4e^a(v_a - u_a)\right) \rho + \pi_{ab} e^a e^b \right]. \end{aligned}$$

Here ρ and π are the energy density and anisotropic stress for the photons only; v_a is the electron 4-velocity, which is the same as that of the baryons. The anisotropic stress entered the expression because of the nontrivial angular dependence of Thomson scattering.

This equation is still somewhat opaque because the angular dependence of $f(E, e)$ is arbitrary. This angular dependence changes with time and couples together all the higher order moments of f. The standard approach in transport theory is to expand this angular dependence in orthogonal polynomials, giving an infinite tower of equations for the moments. This is what we will do. The covariant form of this expansion is actually kind of trivial; we simply write

$$f(E, e) = \sum_{l=0}^{\infty} F^{(l)}_{a_1 \cdots a_l}(E) e^{a_1} \cdots e^{a_l}.$$

Without loss each $F^{(l)}$ is a symmetric tensor and for all $l > 1$ they are traceless (trace parts would just interfere with lower order terms in the expansion and can always be eliminated); by construction $u^a F_{a \cdots b} = 0$. Clearly we have relations such as

$$\rho = 4\pi \int_0^\infty E^3 dE\, F^{(0)},$$
$$q_a = -\frac{4\pi}{3} \int_0^\infty E^3 dE\, F^{(1)}_a,$$
$$\pi_{ab} = \frac{8\pi}{15} \int_0^\infty E^3 dE\, F^{(2)}_{ab}.$$

So the expansion generalizes our decomposition of a stress tensor to the situation where we have a full tower of kinetic theory moments. Defining some energy-integrated quantities,

$$J^{(l)}_{a_a \cdots a_l} = \frac{4\pi(-2)^l (l!)^2}{(2l+1)(2l)!} \int E^3 dE\, F^{(l)}_{a_1 \cdots a_l},$$

the Boltzmann equation for radiation becomes the following infinite tower of equations.

$$\dot\rho + \frac{4}{3}\theta\rho + D^a q_a = 0,$$
$$\dot q_a + \frac{4}{3}\theta q_a + D^b \pi_{ab} + \frac{4}{3}\rho w_a - \frac{1}{3} D_a \rho - n_e \sigma_T \left(\frac{4}{3}(v_a - u_a) - q_a \right) = 0,$$
$$\dot\pi_{ab} + \frac{4}{3}\theta\pi_{ab} + D^c J_{abc} - \frac{2}{5}\left(D_{(a} q_{b)} - \frac{1}{3} h_{ab} D^c q_c \right) - \frac{8}{15}\rho\sigma_{ab} + \frac{9}{10} n_e \sigma_T \pi_{ab} = 0,$$
$$\dot J_{abc} + \frac{4}{3}\theta J_{abc} + D^d J_{abcd} - \frac{3}{7}\left(D_{(a} J_{bc)} - \frac{2}{5} D^d J_{d(a} h_{bc)} \right) + n_e \sigma_T J_{abc} = 0,$$
$$\text{etc.}$$

Given the above hierarchy we can define the temperature variation in different directions. We write the energy density or flux as proportional to the fourth power of the temperature. We can take this as the definition of an effective temperature when the photon distribution deviates from thermal. Let $\delta_T(e)$ be the fractional temperature variation observed in direction e, in the frame given by u. Then

$$(1 + \delta_T(e))^4 = \frac{4\pi}{J^{(0)}} \int E^3 dE\, f(E, e),$$

where

$$J^{(0)} = \int E^3 dE d\Omega(e) f(E, e).$$

Assuming δ_T is small we get a linear expression in terms of the higher moments

$$\delta_T(e) = \frac{1}{4\rho}\sum_{l=1}^{\infty}\frac{(2l+1)(2l)!}{(-2)^l(l!)^2}J^{(l)}_{a_1\cdots a_l}e^{a_1}\cdots e^{a_l}.$$

These are the temperature variations you would see if you sat still and pointed your detector in different directions.

2.4. **Baryon Hydrodynamics.** Hydrodynamics arises from kinetic theory when local equilibrium is a good approximation. In the case of baryonic matter, hydrodynamics is applicable throughout the regimes of interest to us. We could start with kinetic theory for the baryons as well; many calculations of cosmological perturbations have started there. But that would take us too far afield for our purposes. Instead we apply conservation of energy to the coupled photon-baryon fluid, $\nabla^a(T^\gamma_{ab}+T^B_{ab})=0$, and we write the baryon stress energy in fluid form,

$$\begin{aligned}T^B_{ab} &= (\rho+p)v_a v_b - pg_{ab},\\ &= \rho u_a u_b - ph_{ab} + (\rho+p)\left[u_a(v_b-u_b)+u_b(v_a-u_a)+\cdots\right].\end{aligned}$$

The relative velocity $v-u$ is a first-order quantity in cosmological perturbation theory. Let $V^I = v^I - u$ for any species I.

The photon stress tensor is put in the form given by the general decomposition above. As noted, the higher moments do not contribute to the stress tensor, so they will not appear in the baryon evolution equations. Also, no photon moments appear in the baryon energy conservation equation either since the photon-baryon coupling is purely through scattering. We have

$$\dot\rho^B + (\rho^B+p^B)(\theta + D^aV_a) = 0,$$

$$(\rho^B+p^B)(\dot V_a + w_a + \frac{1}{3}\theta V_a) + \dot p^B V_a - D_a p^B + n_e\sigma_T\left(\frac{4}{3}\rho^\gamma V_a - q_a\right) = 0.$$

2.5. **Cold Dark Matter.** Cold dark matter is described as an ideal pressureless fluid. Since cold dark matter is essentially noninteracting, it cannot maintain local equilibrium in the way that a normal fluid does. Therefore, in principle one must keep track of all the correlations in the kinetic theory. However, it turns out that this is not necessary; hydrodynamics is still applicable because the development of higher order correlations in phase space is suppressed in the extreme non-relativistic limit, which is appropriate for CDM particles. Of course, baryons are similarly non-relativistic, but collisions on a short time scale lead to pressure there, as well as to isotropizing effects that also prevent the development of higher order correlations in phase space. The CDM evolution equations are

$$\dot{\rho}^C + \theta\rho^C + \rho^C D^a V_a^C = 0,$$
$$\dot{V}_a^C + w_a + \frac{1}{3}\theta V_a^C = 0.$$

In essence this is the geodesic equation. The forces in the equation are due to the frame choice. If we choose the frame with $u = v^C$, so $V^C = 0$, then the equations become

$$\dot{\rho}^C + \theta\rho^C = 0,$$
$$\dot{v}_a^C = 0.$$

Then the acceleration vanishes, $w = \dot{v}^C = 0$, as it should for geodesic motion. For this and other reasons it is useful to choose this frame, as we will do later.

2.6. **Transition to Free Streaming.** Consider the limit where the density of Thomson scatterers goes to infinity, $n_e \to \infty$. This is a singular perturbation of the tower of equations for the evolution of the radiation and baryons. We will first examine the naive limit for this singular perturbation, without worrying about initial layers or matching conditions. The baryon velocity equation in this limit gives

$$\frac{4}{3}\rho^\gamma(v_a - u_a) - q_a = 0 + \mathcal{O}(n_e^{-1}).$$

This result is easy to understand. Recall that q_a is the heat flux in the frame given by u_a. Since $v_a - u_a$ is the velocity of the baryons in that frame, $\frac{4}{3}\rho^\gamma(v_a - u_a)$ is the heat flux that the baryons see due to relative photon motion, and this lowest order result says that this must equal the total photon heat flux. On other words, if we go to a frame where the baryons are at rest, the heat flux must vanish. This means that the photons are tightly tied to the baryons. It is interesting that this result depends only on the isotropizing effect of Thomson scattering and does not require any actual thermalization process. It holds independent of the nature of the radiation energy spectrum. This result is identical with the result obtained from the photon heat flux equation, so the equations are consistent in this limit, as they must be.

Now examine the higher equations in the photon tower. They all have a similar form, with a covariant time derivative, some spatial gradients, and a term proportional to $n_e\sigma_T$. The singular perturbation of these equations gives $J^{(l)}_{a_a \cdots a_l} = 0 + \mathcal{O}(n_e^{-1})$ for $l \geq 2$. Therefore, in this limit the photons are completely described by an energy density and a heat flux. Therefore, for our practical purposes, this is radiation hydrodynamics. We need only be

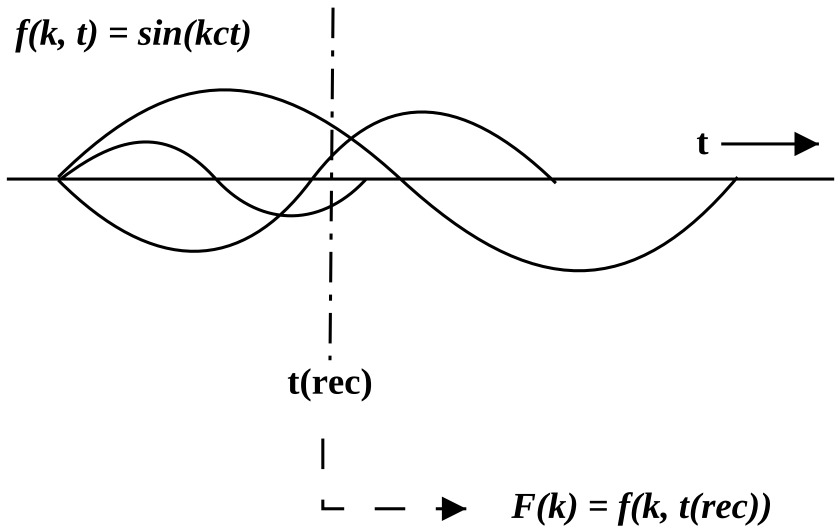


FIGURE 1. Roughly speaking, the radiation carries a snapshot of the state of the gas modes at recombination, bringing that picture forward to the present time. The evolution of the photons between recombination and the present builds up higher angular moments, almost purely by free streaming.

careful to track the true energy distribution of the radiation in circumstances where it is important.

At this point we can construct a picture of what happens to the radiation in our Universe. At early times, the Universe is ionized and the density of scatterers is high. Thomson scattering then effectively couples the photons and baryons and suppresses the growth of higher order moments in the radiation spectrum. This is called the *tightly coupled* regime. At some time, which depends on the local temperature, decoupling occurs. As we learned in the first lecture, this happens right near the time that the ionization fraction drops rapidly (recombination). After this time, the Universe is transparent, with an optical depth of about 10^{-3}, so Thomson scattering suddenly turns off. After this time, the photons begin *free streaming*, building up anisotropic stress and all higher moments. Later we will see how this picture must be corrected to account for damping of perturbations at high wave numbers.

Note that neutrinos, since they are effectively noninteracting at these times, free stream through this era to the present. Because of this, neutrino density and velocity fluctuations are the main source of anisotropic stress.

3. PUTTING THINGS TOGETHER

3.1. **Introduction.** In this section we collect the various forms for the evolution equations and put them in a coherent and useful form. We will eliminate the spatial derivatives by introducing mode functions depending on a wave-number. This will reduce the equations to ODEs for each mode.

3.2. **Modes.** So far we have not said much about the spatial gradients in the equations. In order to write a set of ODEs with which we can compute, we need to eliminate the spatial derivatives by doing some kind of Fourier analysis. Because the spatial sections of the Universe may be curved, it is not appropriate to use plane waves. However, in general terms it is fairly easy to write down the correct form of this expansion for general curved spatial sections.

The idea is to use basis functions of a high degree of symmetry. Complete sets of such functions can be obtained as solutions to the Helmholtz equation in the spatial sections. It is convenient (and geometrically motivated) to use a co-moving definition of the associated eigenvalue (wave-number).

$$D^a D_a Q = \frac{k^2}{S^2} Q.$$

Because of the choice to use a co-moving wave-number, which leads to the factor of S, $\dot{Q}$ is first order in ϵ. When the background is spatially flat the mode functions at zeroth order are $Q(k) = \exp(-ik \cdot x)$.

We also need basis functions for tensorial objects. In general not all of the tensorial objects of interest will be derivable from scalar potentials. But if they are then they can be expanded in terms of derivatives of the scalar functions Q_k. So we introduce

$$\begin{aligned} Q_a &= \frac{S}{k} D_a Q, \\ Q_{ab} &= \frac{S}{k} \left[D_{(a} Q_{b)} - \frac{1}{3} D^c Q_c h_{ab} \right], \\ &\text{etc.} \end{aligned}$$

Then in the spatially flat case

$$\begin{aligned} Q_i &= -i \frac{k_i}{k} \exp(-ik \cdot x) + \mathcal{O}(\epsilon), \\ Q_{ij} &= -S \left[\frac{k_i k_j}{k^2} - \frac{1}{3} \delta_{ij} \right] \exp(-ik \cdot x) + \mathcal{O}(\epsilon), \\ &\text{etc.} \end{aligned}$$

The quantities that appear in the formula for the local temperature anisotropy are $J^{(l)}_{a_1\cdots a_l} e^{a_1}\cdots e^{a_l}$.

3.3. **Evolution Equations.** The evolution equations for the perturbations are simplified if we introduce for each species the following covariant measure of density perturbations.

$$\mathcal{X}_a[\rho] = \frac{S}{k}\frac{D_a\rho}{\rho}.$$

Notice that this is a vector, where you might have expected a scalar expression such as "$\delta\rho/\rho$".

For the rest of our discussion we will consider perturbations without gravity waves. This means that there will be only one physical gravitational degree of freedom, which we might think of as a Newtonian potential. Vector perturbations will not be considered since they die away, as we saw in a previous lecture.

So in the absence of gravity waves and vector modes the gravitational force is derived from a potential; this includes the shear in the 3+1 decomposition. As a matter of definition, when all the tensorial variables of interest can be constructed as derivatives of scalar potentials, we call the associated modes *scalar perturbations*. In this case we use the scalar potentials as fundamental variables. The relations defining them are

$$\begin{aligned}
\mathcal{X}_a(k) \text{ from } \mathcal{X}(k): &\quad \mathcal{X}_a(k) = \int d^3k\, k\mathcal{X}(k)Q_a(k),\\
q_a(k) \text{ from } q(k): &\quad q_a(k) = \rho^\gamma \int d^3k\, q(k)Q_a(k),\\
V_a(k) \text{ from } V(k): &\quad V_a(k) = \int d^3k\, V(k)Q_a(k),\\
\mathcal{Z}_a(k) \text{ from } \mathcal{Z}(k): &\quad \mathcal{Z}_a(k) = \frac{1}{S}\int d^3k\, k^2\mathcal{Z}(k)Q_a(k),\\
\pi_{ab}(k) \text{ from } \pi_{ab}: &\quad \pi_{ab}(k) = \rho^\gamma \int d^3k\, \pi(k)Q_{ab}(k),\\
\text{etc.} &
\end{aligned}$$

Now we can write the evolution equations for the scalar perturbations in terms of these new variables. We introduce the baryonic sound speed by $dp^B = c^2_{sB}d\rho^B$ and the baryonic equation of state $w_B = p^B/\rho^B$. Also, we might as well specialize to the frame choice $u = v^C$; the equations for arbitrary frame choice are given in the references.

$$\dot{\mathcal{X}}^\gamma(k) = -\frac{k}{S}\left(\frac{4}{3}\mathcal{Z}(k) + q^\gamma(k)\right),$$
$$\dot{\mathcal{X}}^B(k) = (1+w_B)\frac{-k}{S}\left(\mathcal{Z}(k) + V^B(k)\right) + (w_B - c_{sB}^2)\theta\mathcal{X}^B(k),$$
$$\dot{q}^\gamma(k) = \frac{1}{3}\frac{k}{S}\left(\mathcal{X}^\gamma(k) - 2\pi^\gamma(k)\right) + n_e\sigma_T\left(\frac{4}{3}V^B(k) - q^\gamma(k)\right),$$
$$\dot{V}^B(k) = (c_{sB}^2 - \frac{1}{3})\theta V^B(k) + \frac{1}{1+w_B}\left(\frac{k}{S}c_{sB}^2\mathcal{X}^B(k) - n_e\sigma_T\frac{\rho^\gamma}{\rho^B}\left(\frac{4}{3}V^B(k) - q^\gamma(k)\right)\right),$$
$$\dot{\pi}^\gamma(k) = -\frac{3}{5}\frac{k}{S}J^3(k) + \frac{2}{5}\frac{k}{S}q^\gamma(k) + \frac{8}{15}\frac{k}{S}\sigma(k) - \frac{9}{10}n_e\sigma_T\pi^\gamma(k),$$
$$\dot{J}^l(k) = -\frac{k}{S}\left(\frac{l+1}{2l+1}J^{l+1}(k) - \frac{l}{2l+1}J^{l-1}(k)\right) - n_e\sigma_T J^l(k),$$

+ equations for neutrinos, CDM, and gravity

4. DAMPING

The story we gave above, about the transition from tight coupling to free streaming, is not quite correct. In taking the scattering term large we assumed that the spatial gradients were bounded, i.e. the wave-number was fixed. But the wave-number can be arbitrarily large. At high enough k, k^{-1} becomes comparable to the mean free path $(n_e\sigma_T)^{-1}$. For such k the assumption of equilibration between photons and baryons is not valid. What this really means is the following.

Suppose we watch an acoustic wave in the coupled system. As some region compresses the radiation there will heat up. The photons start streaming away from this hotter region (at macroscopic scales we call this process heat conduction, where it is a diffusive process). If the wavelength of the compression wave is large, then the photons do not really escape the local region in the time it takes for the wave to cycle, so there is no appreciable loss of energy from the compression. But if the wavelength is small then the photons stream out, they effectively conduct energy out of the compression wave, in the form of heat.

Our goal is to find the rate of mechanical energy loss in high k compressional waves due to this dissipation. The most direct way is to carry the singular perturbation theory, $n_e \to \infty$, to higher orders. It turns out that it is sufficient to go to first order. Let $t_T = (n_e\sigma_T)^{-1}$ be the mean free time. Introduce the quantity $\Delta(k) = q^\gamma(k) - 4V^B(k)/3$, which vanishes in strict tight coupling. Rearranging the dynamical equations to bring factors of t_T

to the numerators gives a set of equations which look schematically like

$$\begin{aligned}(1+R)\Delta(k) &= -t_T F_\Delta(\dot{\Delta}, V^B, \mathcal{X}^\gamma, \mathcal{X}^B, \pi^\gamma),\\ (1+R)\dot{V}^B(k) &= \cdots + \mathcal{O}(t_T),\\ \pi^\gamma &= -\frac{10}{9} t_T F_\pi(\dots),\\ J^l &= -t_T\left[\dot{J}^l + \frac{k}{S}\left(\frac{l+1}{2l+1}J^{l+1} - \frac{l}{2l+1}J^{l-1}\right)\right].\end{aligned}$$

The idea is to iterate these equations giving a formal series in t_T. The first iteration gives

$$\begin{aligned}\Delta &= \frac{t_T}{3(1+R)}\left[\frac{k}{S}(\mathcal{X}^\gamma - 4c_{sB}^2\mathcal{X}^B) + 4HV^B\right],\\ (1+R)\dot{V}^B &= \frac{1}{1+R}\left(\frac{1}{4}\frac{k}{S}R\mathcal{X}^\gamma + \frac{k}{S}c_{sB}^2\mathcal{X}^B - HV^B\right) + \dots,\\ \pi^\gamma &= \frac{16}{27}\frac{t_T k}{S}(\sigma + V^B).\end{aligned}$$

Plugging in a harmonic solution for all the modes, $\exp(-i\omega(k)t)$, gives a dispersion relation of the form

$$\omega(k) = ck + ik^2\lambda_D,$$

where

$$\lambda_D = \pi\sqrt{t_T H^{-1}}.$$

So the modes are in fact damped for wavelengths smaller than λ_D. At the time of recombination we have

$$\begin{aligned}\frac{\lambda_D}{t_{rec}} &\simeq \pi\sqrt{\frac{t_T}{t_{rec}}}\\ &\simeq 0.1.\end{aligned}$$

So this damping wavelength is roughly $1/10$ of the sound horizon, ct_{rec}.

5. Interlude on Boundary Conditions

5.1. **Linearly Independent Solutions.** Pick a particular k mode of the perturbations. At early enough times this mode is super-horizon, meaning $|k\eta| < 1$. On such scales the evolution of the mode is simplified. This is the regime in which to specify the initial conditions. Note that we cannot specify an initial condition at $t = 0$ since this is a singular point of the equations. So we must specify the conditions by giving an appropriate combination of linearly independent solutions, described by their asymptotic behaviours as $t \to 0$.

We have four species of matter: photons, baryons, neutrinos, and CDM. At early times for a fixed wave number, tight coupling holds and we can eliminate the baryon velocity and the photon moments higher than $l = 1$. Also, at early enough times the neutrinos were thermalized and therefore had vanishing higher moments. There are three gravitational degrees of freedom, the shear σ, the expansion gradient $\mathcal{Z}$, and the electric part of the Weyl tensor, which we can identify with a Newtonian potential, Φ. But two of these gravitational degrees of freedom are removed by the constraints. So we apparently have eight degrees of freedom. We can further eliminate the CDM velocity by choosing the frame $u = v^C$. Finally we can eliminate the CDM and baryon densities since they are dynamically irrelevant at early times; this is chiefly to simplify the discussion. That leaves five degrees of freedom.

Examining the equations for $t \to 0$ shows that of the five possible independent modes, two are decaying. As discussed in previous lectures, we ignore decaying modes. That leaves three dominant modes. Of these, the most important is the adiabatic mode. Adiabaticity means that density variations of all the species are tied together by relations of the form

$$\frac{Ds^i}{s^i} = \frac{D\rho^i}{\rho^i + p^i} = \frac{D\rho^j}{\rho^j + p^j} = \frac{Ds^j}{s^j}.$$

In other words, the matter moves together in such a way as to tie together changes in specific entropy.

The other modes allow the densities of different species to vary independently. These modes are like second sound; they are entropy waves. When the density variations are perfectly anti-correlated there are no variations in the gravitational potential and such modes are pure entropy waves. In the literature these are called isocurvature modes.

5.2. **Inflation.** Of course, the equations above give a completely deterministic picture for the evolution of cosmological perturbations. But what should we take for the initial conditions? The honest answer is that nobody really knows. Three issues arise when considering the initial conditions. First, the Universe appears to be very nearly homogeneous and isotropic on large scales. Second, the standard cosmologies cannot provide a causal explanation for this large-scale smoothness, because all such cosmologies have horizons. To see this note that $\Delta\theta = \Delta L/d_A(z)$, where $d_A(z)$ is the angular diameter distance to redshift z. We have

$$
\begin{aligned}
\Delta\theta &= \frac{H_0^{-1}(1+z)^{-3/2}}{2H_0^{-1}(1+z)^{-1}} \\
&\simeq \frac{1}{2(1+z)^{1/2}} \\
&\simeq 1^o \left(\frac{1000}{1+z}\right)^{1/2} .
\end{aligned}
$$

Therefore, in our Universe, the horizon at the time of recombination corresponds to a patch of sky with an angular size of about 1 degree. The third point is perhaps somewhat philosophical (but only somewhat). Since the Universe is filled with apparently random fluctuations, what is the source of this apparent randomness? A stochastic theory of the initial conditions seems like the only appealing way to arrive at such a state.

Thus far, the only acceptable picture for the initial state of the Universe comes out of the ”inflationary universe paradigm”. Like many cosmologists, I hesitate to call inflation a theory, since it seems to encompass a large class of cosmologies, all of which successfully address the issues raised above. Inflation will be considered in a separate lecture, later in the series. Briefly, it has the following properties.

In an inflationary universe, the whole of the visible Universe arises as a tiny equilibrated patch in a larger universe. Either this larger universe, or perhaps just our patch, suffered some odd ”superluminal” expansion, so that the visible Universe could grow to an appropriate size. If such an expansion can get started (the hard part), then it leads naturally to a smooth visible universe. Furthermore, quantum fluctuations are expanded and frozen as classical fluctuations, leading to a stochastic source for the cosmological perturbations needed to seed large scale structure.

A typical property of these initial fluctuations, although by no means absolutely required, is that they are Gaussian. Gaussianity of the initial fluctuations remains a subject for argument and may be resolved by measurement of correlations in the microwave sky. Another typical feature of these fluctuations is that they are adiabatic. As we saw above, adiabatic fluctuations are characterized by a single amplitude as a function of wave-number. The statistical properties of such a single Gaussian random field are specified by a single function $P(k)$, the power spectrum. By predicting this power spectrum, inflation predicts the stochastic properties of cosmological perturbations.

So, our best theoretical straw-man is that the initial conditions for our Universe correspond to a realization of a Gaussian random field. For the rest of this lecture, when this input is needed, we will assume that the initial perturbations are described by such an ensemble. Of course, this means that our predictions are reduced to statistical predictions, either implicitly or explicitly. This also satisfies our desire for a stochastic theory of the initial conditions.

6. Line of Sight Integral

Recall the expression for the temperature fluctuation in terms of the $J^{(l)}_{a_1\cdots a_l}$,

$$\begin{aligned}\delta_T(e) &= \frac{1}{4\rho^\gamma}\sum_{l=1}^{\infty}\frac{(2l+1)(2l)!}{(-2)^l(l!)^2}J^{(l)}_{a_1\cdots a_l}e^{a_1}\cdots e^{a_l},\\ &= \frac{1}{4\rho^\gamma}\sum_{l=1}^{\infty}\frac{(2l+1)(2l)!}{(-2)^l(l!)^2}J^{l}Q_{a_1\cdots a_l}e^{a_1}\cdots e^{a_l}.\end{aligned}$$

In the second equation we have used the defining expansion for $J^{(l)}_{a_1\cdots a_l}$ in terms of J^l.

In principle we need to calculate the evolution of all the quantities in perturbation theory, and thus obtain the solution for the J^l, which we can use to obtain the temperature fluctuation. There is no way to avoid the solution of the fully coupled system in the numerical calculation of δ_T. However, it is possible to obtain an intermediate result which gives a solution for δ_T as an integral representation given the time evolution of other quantities. There are different ways to write this integral representation, but the most useful is the so-called line-of-sight form.

That the radiation hierarchy should admit such a solution is a little surprising at first. But when one realizes that the main effect of the evolution is to geometrically project the state at early times onto the state at late times it becomes less surprising. In fact, this is really the full content of the free streaming solutions in the absence of coupling; they simply move the particles through phase space, and the development of anisotropy is a statement about the geometry of this projection. We will look briefly at this free streaming behaviour now.

Consider the equations for the higher moments, $J^{(l)}$, $l \geq 3$. For the sake of our argument, assume that the scattering terms all vanish. Then the equations for these moments look like

$$\dot{J}^l + \frac{k}{S}\left[\frac{l+1}{2l+1}J^{l+1} - \frac{l}{2l+1}J^{l-1}\right] = 0.$$

Introduce a scaled conformal time variable $\tilde{y} = k\eta = k \int dt S^{-1}(t)$, then the scale factor can be removed from the equation,

$$\frac{d}{d\tilde{y}} J^l + \left[\frac{l+1}{2l+1} J^{l+1} - \frac{l}{2l+1} J^{l-1} \right] = 0.$$

But this equation is equivalent to a recurrence relation for spherical Bessel functions; with an appropriate choice of initial conditions the solution is $J^l = j_l(k\eta)$. This result is not surprising since the equations in this form describe the pure free-streaming of radiation, so the solution should be the radial solution of the wave equation.

Now suppose that we put back the scattering terms,

$$\dot{J}^l + \frac{k}{S} \left[\frac{l+1}{2l+1} J^{l+1} - \frac{l}{2l+1} J^{l-1} \right] = -n_e \sigma_T J^l.$$

Again this has a simple solution, which we can get by guessing if necessary,

$$J^l = \exp\left(-\int n_e \sigma_T \right) j_l(\tilde{y}).$$

The exponential suppression is what one could guess, knowing that $\int n_e \sigma_T$ is related to the optical depth.

The actual system is complicated by the coupling between multipoles due to the scattering terms, which forces the solution to contain higher derivatives of basis functions and makes it impossible to fully integrate the result.

The final form for the line-of-sight integral is

$$\begin{aligned} \delta_T(e)_0 = &-\frac{1}{4} \int d^3k \, (\mathcal{X}^\gamma Q(k))_0 \\ &+ \int d^3k \int^{t_0} e^{-\tau} \left(\frac{k}{S} \sigma(k) e^a e^b Q_a b(k) - \frac{1}{3} \frac{k}{S} \mathcal{Z}(k) Q(k) \right) \\ &+ \int d^3k \int^{t_0} \kappa \left(\frac{3}{16} \pi^\gamma(k) e^a e^b Q_{ab}(k) - V^B(k) e^a Q_a(k) + \frac{1}{4} \mathcal{X}^\gamma(k) Q(k) \right), \end{aligned}$$

where we have introduced the visibility function $\kappa = -\tau e^{-\tau}$. The visibility function is peaked near the point where the optical depth drops, so the last term gives a contribution localized to that time. As is often stated in the literature, the best way to see that this is a solution is to plug it into the equation for δ_T which follows from the hierarchy. We will not bother to check it explicitly here.

7. Temperature Fluctuations

7.1. **Simplified Line-of-Sight Integral.** As noted above, In order to actually calculate the anisotropy, we need to solve the evolution equations for the quantities that appear in the line-of-sight integral and then evaluate it. We cannot do this here, but what we can do is understand how the various terms in that expression contribute to the temperature anisotropy.

The first thing we can do is make the approximation that recombination happens instantaneously. This makes sense for wave numbers which are not too large; we demand that the frequency of the wave is low compared to the inverse time scale for scattering. For our Universe this corresponds to angular scales on the current sky which are larger than about $8'$, which is consistent with the damping time scale that we calculated previously. So we assume the optical depth τ drops abruptly from infinity to zero at t_{rec}. The visibility function κ is then a delta function, so the last term in the line-of-sight integral becomes completely local. Part of the contribution from this cancels the local term from t_0 in the expression. After some algebra we are left with

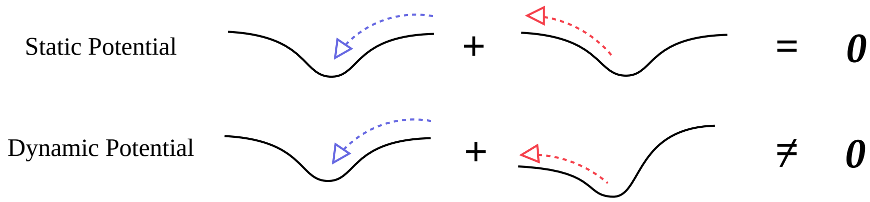
$$\begin{aligned}
\delta_T(e)_0 \simeq &\int d^3k \, \left[\left(\frac{1}{4}\mathcal{X}^\gamma(k) + \frac{S}{k}\dot\sigma(k)\right) Q(k)\right]_{rec} \\
&- \int d^3k \, \left[(V^B(k) + \sigma(k))e^a Q_a(k)\right]_{rec} \\
&+ \frac{3}{16}\int d^3k \, \left[\pi^\gamma(k) e^a e^b Q_{ab}(k)\right]_{rec} \\
&- 2\int d^3k \int^{\eta_0} \dot\Phi(k) Q(k).
\end{aligned}$$

The Einstein equations have been used to replace some terms with the expression involving $\dot\Phi$, where Φ is the scalar potential for the electric part of the Weyl tensor. Think of Φ as a covariant form of the Newtonian potential.

The first two terms with local contributions from last scattering are called the monopole and dipole terms. Because the photon-baryon fluid was tightly coupled before last scattering, these are the only contributions; higher moments could only become nonzero after recombination. In a rough sense the monopole term represents everything that has to do with the state of the fluid on the last scattering surface, and the dipole term has to do with motion. Note the explicit appearance of the baryon velocity; this term represents a Doppler shift in the temperature due to fluid motion just at the point of last scattering.

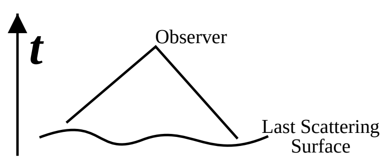
The term proportional π^γ vanishes in the exact tight coupling limit, but we have left it because actual calculations will calculate the next to leading order tight coupling evolution.

The final term is an integrated type of Sachs-Wolfe effect, sometimes called the Rees-Sciama effect. The idea is that time variation in the gravitational



potential can lead to an isotropy because of the non-vanishing time it takes for a photon to fall into and then climb out of a potential well. During this time the potential can change, leading to a net frequency shift. In flat cosmologies it turns out that this term gives a only a small contribution.

7.2. **Monopole Contribution.** The monopole term includes the intrinsic temperature variation and an effect proportional to the gravitational potential at that the time of last scattering. We might think that the intrinsic variation corresponds to temperature variation on the surface of last scattering, but this is not correct. If the ambient gas temperature along some line of sight were intrinsically higher, decoupling would happen later, but it could not happen at a different temperature. So it would not be correct to say that this term *directly* tracks fluctuations in the intrinsic background temperature. Instead, it represents a geometric effect; the surface of last scattering (to the extent that it is a well-defined surface) is not embedded at a fixed time but should be drawn wrinkled in a spacetime diagram.



This wrinkling leads to a variation in the redshift. The local nature of this contribution arises because of the instant recombination approximation.

The effect proportional to the gravitational potential is less easy to extract. Applying one of the Einstein constraint equations allows us to write

$$\frac{S}{k}\dot{\sigma}(k) = -\frac{1}{3}\Phi(k) + \text{other terms},$$

where as before Φ is the potential for the electric part of the Weyl tensor, which we can think of as the Newtonian potential. This gravitational contribution is called the Sachs-Wolfe effect. Note the famous factor of $1/3$. You might expect a factor of unity due to a simple redshift, but there is partially compensating time-delay effect; the extra factor of $-2/3$ comes from the scaling $a(t) \propto t^{2/3}$ during matter domination.

7.3. **Dipole Contributions.** The dipole contribution proportional to the baryon velocity is easy to understand as a Doppler effect, as we have already noted. The gravitational contribution comes because variations in the potential cause peculiar motions; a dynamical effect. The size of a region effected by an acoustic wave should be less than the acoustic horizon, which is $\sqrt{1/3}$ times the light horizon. Again, this is about one degree on the sky, as we estimated above.

7.4. **Multipole Expansion.** The observational meaning of a temperature variation is obvious. Radiation is collected (time-integrated) along a line of sight; the spectrum is assigned a temperature. This is reasonable since we know it is very close to thermal, ignoring for the moment any contamination issues. Measuring along different lines of sight, we can make a temperature map on the sky. We have already seen the multipole expansion for the temperature fluctuation, in its covariant form. The spacelike vector e specifying the direction of the radiation can be written in the u frame as

$$e^\mu = (0, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Then we have

$$\delta_T(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\theta,\phi),$$

where

$$\frac{1}{2l+1}\sum_{m=-l}^{l} a_{lm}Y_lm(\theta,\phi) = \frac{1}{4\rho^\gamma}\frac{(2l)!}{(-2)^l(l!)^2}J^{(l)}_{a_1\cdots a_l}e^{a_1}\cdots e^{a_l}.$$

Most treatments of the phenomenology of temperature anisotropy measurements use such a spherical harmonic expansion on the sky.

We can define a quantity quadratic in the harmonic amplitudes,

$$c_l = \frac{1}{2l+1}\sum_{m=-l}^{l}|a_{lm}|^2.$$

The quantity $(2l+1)c_l$ has the interpretation of the power contained on scales sampled by the l modes. In the last lecture we will see how this quantity is is interpreted given our stochastic picture of the initial conditions.

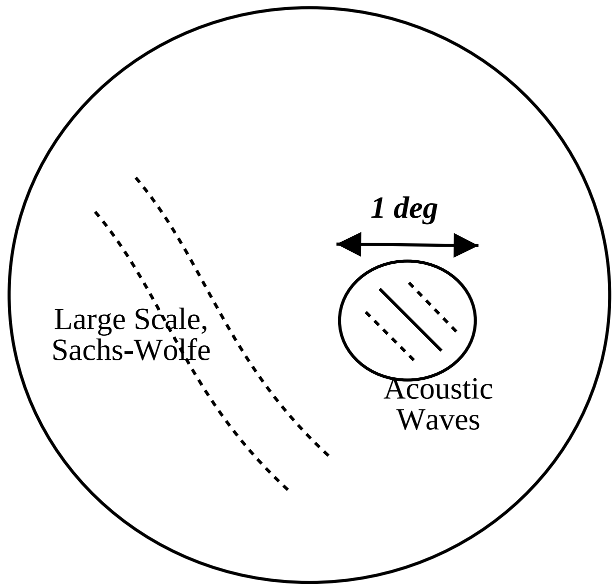


FIGURE 2. A cartoon of mode excitations on the last scattering surface, as seen on a piece of the current sky. The Sachs-Wolfe effect leads to variations at large scales. Acoustic oscillations concentrate power in modes with smaller wavelengths, within the $\simeq 1^o$ acoustic horizon.

7.5. **Multipole Power Spectrum.** For now we will imagine initial conditions which correspond to some spectrum of fluctuations. The freedom in initial conditions is then specified by mode occupation function; the simplest choice is a power-law spectrum in wave-number k. Theoretical prejudice, including inflationary predictions, point to a scale-invariant power spectrum over a large range of k. These initial conditions are processed by all the dynamics that we have described, so that power is shifted about, becoming concentrated at scales corresponding to acoustic oscillations and being distributed among larger scales by the Sachs-Wolfe effect.

The photons take a picture of this state of affairs around the time of recombination. Free-streaming builds up occupation in higher radiation multipoles, giving the pattern on the sky which we see. Expanding that pattern in multipoles and calculating the c_l coefficients gives the multipole power spectrum. An example set of such power spectra is given in the final figure. The long flat part of the spectrum shows the contribution from the largest scales, which is mainly the Sachs-Wolfe effect. The first peak occurs at $l \simeq 200$, which corresponds to our acoustic horizon size of about 1^o. This peak reflects the power concentrated in the fundamental ringing mode of

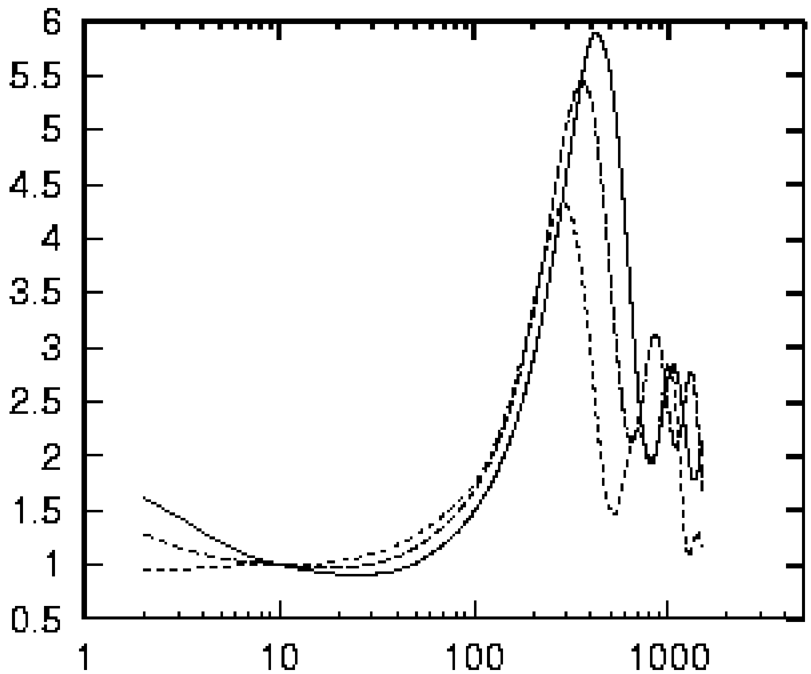


FIGURE 3. Obligatory example of CMB multipole power spectra. Courtesy CMBFast.

the Universe. The higher modes correspond to harmonics at successively higher k and therefore smaller scales.

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